Algebraic topology from the views of applications—Lecture 1

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Algebraic Topology from the Views of Applications—Lecture 1

Preface on the lectures

- There are a lot of ideas from our unified topological approach, which is mainly handled by Kelin now, that can be used for applications to data analysis on complicated networks.

- These lectures aim to “mathematics-mining” that can be used for **potential** applications in future.

Feature of Algebraic Topology

代数拓扑的“三位一体特性”

- 代数拓扑具有如下化身:
  - 形的表现——polyhedron, manifold, cell complex, mapping spaces, configuration spaces, group actions and fibre bundles, fibrations, fibrewise topology, diagrams of spaces···
  - 组合的表现——simplicial complex, Δ-set (semi-simplicial set), simplicial set, nerve complex, cubic complex, simplicial homotopy theory, combinatorial topology···
  - 范畴的表现——nerve of category, closed model category, triangulated category···.
- 代数拓扑用（代数）量去刻画研究对象（objects)在形变（up to deformation)下的不变量（invariants): 数（Euler/Betti number)、多项式（Poincaré series)、群(fundamental group、homology group、homotopy group)、环(cup product)、模(Steenrod module)、多层次结构(spectral sequence)···

Many features of homotopy/deformation/purtubation

- The usual/typical homotopy—Geometric Version:
  - A parametrized continuous map $f_t\colon X \to Y$, $0 \leq t \leq 1$, which continuously depends on t, as a deformation from f_0 to f_1.
  - **homotopy relative to a subspace** A. In this case, $f_0|_A = f_1|_A$, and the homotopy f_t is required to be stationary on A, i.e., $f_t(a) = f_0(a)$ for $a \in A$, $0 \leq t \leq 1$.
  - **set of homotopy classes:** $[X, Y]$ usually refers to $[(X, *_X), (Y, *_Y)]$ for pointed spaces X and Y.
  - A **fundamental/core problem** in Alegrbaic Topology is to determine $[X, Y]$, including the homotopy groups. (**Homology** usually is a tool for studing **homotopy**.)

Many features of homotopy/deformation/purtubation

- The usual/typical homotopy—Algebraic Version:
  - A chain homotopy $F\colon C_* \to D_*$ of degree $+1$ with the rule

$$f - g = F \circ \partial + \partial \circ F$$

  - **More structural objects**: DGA (**differential graded algebras**), **differential graded Lie algebras/Poisson algebras** ..., the chains/cochains with more algebraic structures.
  - **spaces up to rational homotopy** can be determined by DGA (Sullivan model) or differential graded Lie algebras (Quillen model).
  - **Integrally or *p*-locally**, spaces are **far away** from algebras.

Many features of homotopy/deformation/purtubation

- The usual/typical homotopy—Simplicial version:
  - A simplicial map $F\colon X \times \Delta[1] \to Y$ between simplicial sets.

The notion of **simplicial set** is developed from the notion of simplicial complex, with adding "**degenerate/singular simplices**".

It seems that simplicial set may be a good model for simplices with their vertices given by multi-set, such as $\{a, a, b\}$ is treated as a (degenerate/singular) 2-simplex.

The notion of simplicial set is good for theoretical study with having many good properties.

Many features of homotopy/deformation/purtubation

- The usual/typical homotopy—Categorical Version:
    - Categorical version: nerve of category gives a functor $\mathcal{N}\mathcal{C}$ from small categories $\mathcal{C}$ to simplicial sets, functors between categories being mapped to simplicial maps, and natural transformations being mapped to homotopy.

If we view a category $\mathcal{C}$ as a **quiver** with a composition operation on arrows, then $\mathcal{N}\mathcal{C}$ is the **path complex.**

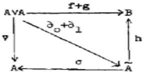
Category may be views as a **quiver (multi-digraph) with dynamics.** Composition may be considered as a rule of arrow-moving.

Many features of homotopy/deformation/purtubation

The non-typical homotopy—Quillen's homotopy[1]:

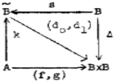
Definition 3: Let $f,g: A \rightrightarrows B$ be maps. We say that f is left-homotopic to g (notation $f \overset{l}{\sim} g$) if there is a diagram of the form

(3)



where σ is a weak equivalence. Dually we say that g is right-homotopic to g (notation: $f \overset{r}{\sim} g$) if there is a diagram of the form

(4)



where s is a weak equivalence.

[1] D. G. Quillen, homotopical Algebra, Lecture Notes in Mathematics **43**, 1967.

Remarks on Quillen's closed model category

Assume that we consider all possible "**working data**" as a category: objects togerther with **morphisms** given by certain rules. There are three family of morphisms under Quillen's consideration in his model category:

- **weak equivalence**: one data is **similar** to another.
- **cofibrantion**: a collection of special morphisms.
- **fibration**: a collection of another type of special morphisms.
- They satisfy some axioms.

Then we try to **classify** our working data up to **weak equivalence** (without computing). (Maybe too hard to do computing.) Mathematically, one can do localization with respect to weak equivalence.

Many features of homotopy/deformation/purtubation

Non-typical version of “homotopy”: Gerstenhaber’s deformation theory of algebras, introduced in 1964.

- Deform a multiplication $A \otimes A \to A$ to $A[t]$, the polynomials on t with coeficients in A.
- Have connections with rational homotopy theory.
- Related to Hochschild cohomology.

- Applications in physics, (algebras associated with) classical mechanics (and field theory) on a Poisson phase space can be deformed to (algebras associated with) quantum mechanics (and quantum field theory).

Many features of homotopy/deformation/purtubation

More robust version of homotopy: **Ambient isotopy** of knots and links

- A paramitrized map $\phi_t \colon S^3 \to S^3$, $0 \leq t \leq 1$, such that
  - ϕ_t smoothly depends on t,
  - ϕ_t is a deffeomorphism for each t,
  - ϕ_0 is the identity, and
  - ϕ_1 sends one knot/link to another.

We may replace knot by a ribbon or the **shape** of molecular with **more constraint hypothesis** on ambient isotopy which only allow isotopy to be carried within a small region.

Non-typical version of “homotopy”—bordism theory, a generalized homology theory

A **singular n-dimensional (sub) manifold** of a space X is a pair (M^n, f), where M^n is a closed smooth manifold, and $f\colon M^n \to X$ is a continuous mapping.
Two such pairs (M^n, f), (N^n, g) are **bordant** if M and N are bordant in the ordinary sense, i.e there exists a smooth compact $(n+1)$-dimensional manifold W(a “membrane”) with its boundary diffeomorphic to the disjoint union of M and N under certain diffeomorphisms, with a continuous mapping $h\colon W \to X$ such that h restricted to the boundary is given by f and g.

A **topological quantum field theory** is a symmetric monoidal functor from the bordism category with ξ-structure to the category of (usual or $\mathbb{Z}/2$-graded) vector spaces over $\mathbb{C}$ with tensor products.

Definition of Homotopy Group

From the definition, $\pi_n(X) = [(S^n, *), (X, *)]$, the set of homotopy classes of base-point preserving continuous maps from the n-sphere S^n to space X.

Base-point preserving is required so that $\pi_n(X)$ is a group for $n \geq 1$.

We call a (continuous) map $f\colon S^n \to X$ a **singular *n*-hole** in X.

Hopf map gives a singular 3-dimen'l hole in a 2-dimen'l sphere S^2

$\pi_3(S^2) = \mathbb{Z}$ generated by the Hopf map $H\colon S^3 \to S^2$.

$\pi_{n+1}(S^n) = \mathbb{Z}/2$ for $n \geq 3$.

Intuitively, **dimension reduction** creates higher dimensional singular holes because of complicated folding.

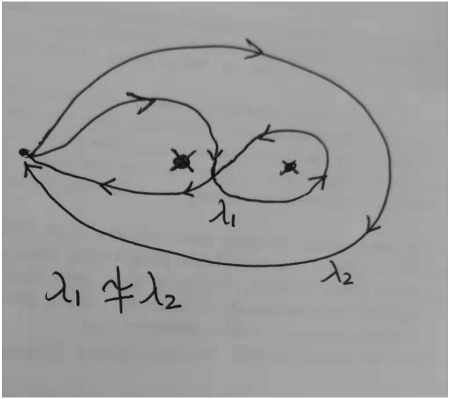
$\pi_*(S^2 \vee S^2)$ contains all higher homotopy groups of any S^n (with $n \geq 2$) as summands—Two dimen'l world contains the universe.

Actually, 2-dimen'l manifolds are shaped **too elegant** to encode higher dimen'l data, such as homotopy groups of higher dimen'l spheres.

The **core/fundamental problem** in algebraic topology: Understanding the structure/Computing $\pi_*(X)$.

Fundamental group $\pi_1(X)$

For a space X, the fundamental group X is the group of homotopy classes of continuous loops (paths starting and ending at the basepoint).



Simplicial Fundamental Groups $\pi_1(K)$

Let K be a simplicial complex. Let v_0 be a vertex treated as the bassepoint. The fundamental group $\pi_1(|K|)$ can be described as the walks/paths (which allows repeating on vertices) starting and ending at v_0 subject to the ewqivalence relation generated by

- $\cdots aa \cdots \sim \cdots a \cdots$, and
- $\cdots abc \cdots \sim \cdots ac \cdots$ if $\{a, b, c\}$ forms a simplex in K

Simplicial fundamental groupoid: objects are vertices of K, and morphisms from v to w are all walks/paths from v to w subject to the above equivalence relation.

Relation between fundamental group and the first homology group

Poincaré Theorem. $H_1(X)$ is the abelianization of $\pi_1(X)$.

Poincaré theorem tells that

- H_1 can detect **cycles** in a data.
- The commutator subgroup $[\pi_1(X), \pi_1(X)]$ in $\pi_1(X)$, generated by commutators $[a, b] = a^{-1}b^{-1}ab$, can not be detected by $H_1(X)$.
- Using mod 2 lower central series of $\pi_1(X)$, one may get more accurate information on cycles.
- Applying to the fundamental groupoid, $H_1(X)$ can be used to detect some information of walks from v to w, which indicates a connection **between H_1 and radom walks**.

Singular *n*-holes to solid *n*-holes—Sphere Theorem

Papakyriakopoulos Theorem (Sphere Theorem). a compact 3-manifold with nontrivial π_2 has a two-sided embedded sphere or projective plane representing a nontrivial homotopy class.

Loop Theorem. the induced map $\pi_1(\partial M) \to \pi_1(M)$ is NOT injective, then an essential loop in ∂M,which is null homotopic in M, that can be represented by an embedded disk in M.

Algebraic n-holes to singular n-holes—Hurewicz Homomorphism

Hurewicz Homomorphism $H\colon \pi_n(X) \to H_n(X)$: For $[f] \in \pi_n(X)$ with $f\colon S^n \to X$, $f_*\colon H_n(S^n) = \mathbb{Z} \to H_n(X)$. Define $H([f]) = f_*(1) \in H_n(X)$.

Hurewicz Theorem. Suppose that $\pi_j(X) = 0$ for $0 \leq j \leq n-1$. Then $H\colon \pi_n(X) \to H_n(X)$ is an isomorphism.

We call cycles in chains $C_*(X)$ **algebraic** ***n*****-holes.** Truely speaking, an algebraic n-hole ω induces a singular n-hole iff $[\omega] \in H(\pi_n(X))$, the Hurewicz image.

Spherical classes

The homology classes in integral/mod p homology lying in the image of Hurewicz image are called **spherical classes**.

Proposition. If $\alpha \in H_*(X)$ is a spherical class, then, in mod p homology,

- α is primive, i.e. in the kernal of $\tilde{H}_*(X) \to \tilde{H}_*(X) \otimes \tilde{H}_*(X)$ induced by reduced diagonal.
- The Steenrod operations acts trivially on α.

In term of cohomology, we can look at 1) indecomposables $\tilde{H}^*(X)/(\tilde{H}^*(X) \cup \tilde{H}^*(X)$, and 2) $\tilde{H}^*(X)/I\mathcal{A} \cdot \tilde{H}^*(X)$, where $I\mathcal{A}$ is the augmentation ideal of the Steenrod algebra.

We will get new persistent modules (and so new topological barcodes) as **further approximation** to n-holes.

Singular grids and homotopy groups

Let K be a cubic complex. Consider S^n as the boundary ∂I^{n+1} of $(n+1)$-dimen'l cube I^{n+1}. Let C_t^n be the subdivision of ∂I^{n+1} in the standard way by cutting it as little cubes. From the lines of **simplicial approximation theorem**, any continuous map $f\colon S^n \to |K|$ can be approximated by a cubic map $g\colon C_t^n \to K$ for some t.

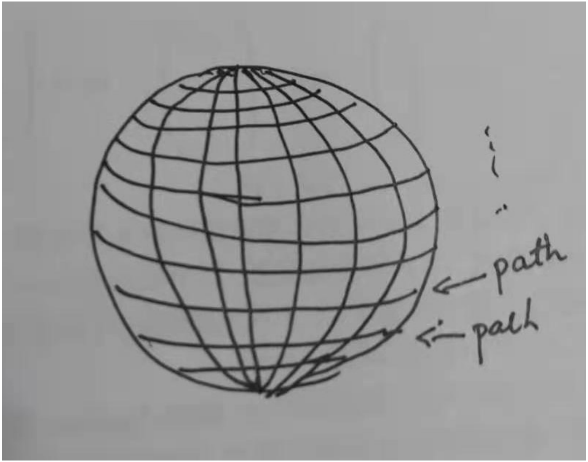
We can try to use Relative Simplicial Approximation Theorem to set up the homotopy of cubic approximations.

We can call a cubic map $g\colon C_t^n \to K$ a **singular grid**.

It seems that we can finally set up a theory on singular grids in networks using homotopy groups.

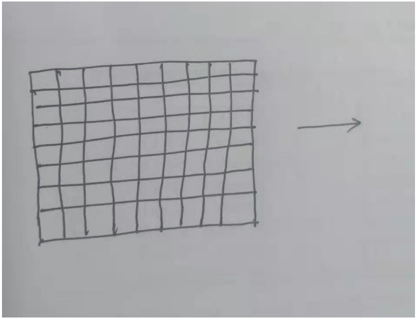
Path between paths

We can consider a map $S^2 \to K$ as a path between paths



More concrete descriptions on path-between-paths using homotopy as desciptor

We may extend the ideas of path homotopy and fundamental groupoids: Let $n = 2$ for simplicity. Consider $f, g \colon I^2 \to X$ with $f|_{\partial I^2} = g|_{\partial I^2}$, and then consider homotopy **relative to the boundary** ∂I^2.



Multi-layer network

It seems that homotopy interpreted as iterated path-between-paths could be a good descriptor for multi-layer network.

$\pi_1(X)$ is well described in simplicial way.

$\pi_n(X)$, $n \geq 2$, are not well-explored in the way what I have described as cubic/simplicial maps or path-between-paths.

Historically, in simplicial homotopy theory, for $\pi_n(K)$, people make the domain S^n as simple as possible and make K to have more elements for catching homotopy.

Thank You for Your Attention!