An introduction to orbifolds

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4. 2000s, Weimin Chen, Yongbin Ruan studied the homotopy groups, cohomology ring of orbifolds . . .

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(2) The collection of all orbifold charts $\{(V_i, \phi_i, U_i, G_i)\}$, called the orbifold atlas, satisfies the condition that for each inclusion $U_i \hookrightarrow U_j$, there exists an injective group homomorphism $g_{ij} : G_i \longrightarrow G_j$ and a G_i-equivariant embedding $\varphi_{ij} : V_i \longrightarrow V_j$ such that $\phi_j \circ \varphi_{ij} = \phi_i$ and φ_{ij} is unique up to composition with elements of G_j.

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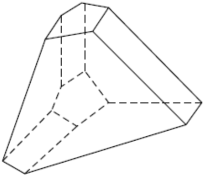

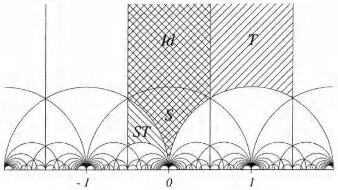

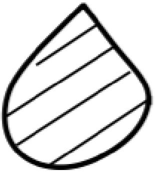

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Examples of good orbifolds

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2. Classic configuration space $F(X,n) = \{(x_1, \cdots, x_n) \in X^{\times n} | x_i \neq x_j \textit{ if } i \neq j\}$, symmetric group Σ_n acts on $F(X,n)$ freely.

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   As to generalization of configuration spaces, such as chromatic configuration space $F(X, \Gamma)$, orbit configuration spaces $F_G(X, n)$.
   $Aut(\Gamma)$ on $F(X, \Gamma)$ and $G^{\times n} \rtimes \Sigma_n$ on $F_G(X, n)$ non free.

A **covering** of a smooth orbifold $\mathcal{O}$ is a pair $(\widehat{\mathcal{O}}, \rho)$, where $\widehat{\mathcal{O}}$ is another orbifold and $\rho : \widehat{\mathcal{O}} \longrightarrow \mathcal{O}$ is a surjective smooth map. There exists a representing system $(\{V_\alpha\}, \{U_\alpha\}, \{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ satisfying that

(1) For any $U \in \{U_\alpha\}$, the set of connected components of $\pi^{-1}(U)$ is contained in $\{V_\alpha\}$;

(2) Each homomorphism $\rho_\alpha : G_{V_\alpha} \longrightarrow G_{U_\alpha}$ is monomorphic, and each $\pi_\alpha : \widehat{V_\alpha} \longrightarrow \widehat{U_\alpha}$ is a ρ_α-equivariant homeomorphism.

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Theorem

Any connected orbifold $\widehat{\mathcal{O}}$ admits a connected universal covering $\rho : \widehat{\mathcal{O}} \longrightarrow \mathcal{O}$.

There are several definition for orbifold fundamental group.

(1) Consider the based loop space of $(\mathcal{O}, \underline{o})$ consisting of equivalence classes of loops in $\mathcal{O}$, where $\underline{o}$ is non-singular base point .
For any $k \geq 1$, the k-th homotopy group of $(\mathcal{O}, \underline{o})$, denoted by $\pi_k^{orb}(\mathcal{O}, \underline{o})$ is defined to be the $(k-1)$-th homotopy group of the based loop space $(\Omega(\mathcal{O}, \underline{o}), \tilde{o})$.

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(3) For a good orbifold, i.e. $G \hookrightarrow Y \longrightarrow Y/G$, consider the Borel construction $Y_G \cong Y \times_G EG$, then $\pi_k^{orb}(Y/G) \cong \pi_k(Y_G)$.

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If P^2 is 4-gon, $\pi_1^{orb}(P^2)$ is

$$\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1, ab = ba, ad = da, bc = cb, cd = dc \rangle$$

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