An introduction to orbifolds

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4. 2000s, Weimin Chen, Yongbin Ruan studied the homotopy groups, cohomology ring of orbifolds . . .
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1. Each open set $U_i$ is associated with an orbifold chart $(V_i, \phi_i, U_i, G_i)$, where $G_i$ is a finite group, $V_i \subset \mathbb{R}^n$ is an open subset that is invariant under an effective action of $G_i$ on $\mathbb{R}^n$, and $\phi_i : V_i \longrightarrow U_i$ is surjective and induces a homeomorphism $V_i/G_i \longrightarrow U_i$. 
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(2) The collection of all orbifold charts $\{(V_i, \phi_i, U_i, G_i)\}$, called the orbifold atlas, satisfies the condition that for each inclusion $U_i \hookrightarrow U_j$, there exists an injective group homomorphism $g_{ij} : G_i \rightarrow G_j$ and a $G_i$-equivariant embedding $\varphi_{ij} : V_i \rightarrow V_j$ such that $\phi_j \circ \varphi_{ij} = \phi_i$ and $\varphi_{ij}$ is unique up to composition with elements of $G_j$. 

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[Diagram of orbifolds]
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Examples of good orbifolds

For a small cover $\pi : M^n \rightarrow P^n$, locally modelled by $\mathbb{R}^n/\mathbb{Z}_2^n$, $P^n \cong M^n/\mathbb{Z}_2^n$ is a good orbifold.
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2. Classic configuration space

$F(X, n) = \{(x_1, \ldots, x_n) \in X^{\times n}| x_i \neq x_j \text{ if } i \neq j\}$, symmetric group $\Sigma_n$ acts on $F(X, n)$ freely.
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As to generalization of configuration spaces, such as chromatic configuration space $F(X, \Gamma)$, orbit configuration spaces $F_G(X, n)$.

$\text{Aut}(\Gamma)$ on $F(X, \Gamma)$ and $G^\times n \rtimes \Sigma_n$ on $F_G(X, n)$ non free.
A covering of a smooth orbifold $\mathcal{O}$ is a pair $(\hat{\mathcal{O}}, \rho)$, where $\hat{\mathcal{O}}$ is another orbifold and $\rho : \hat{\mathcal{O}} \longrightarrow \mathcal{O}$ is a surjective smooth map. There exists a representing system $((\{V_\alpha\}, \{U_\alpha\}, \{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ satisfying that

1. For any $U \in \{U_\alpha\}$, the set of connected components of $\pi^{-1}(U)$ is contained in $\{V_\alpha\}$;
2. Each homomorphism $\rho_\alpha : G_{V_\alpha} \longrightarrow G_{U_\alpha}$ is monomorphic, and each $\pi_\alpha : \hat{V}_\alpha \longrightarrow \hat{U}_\alpha$ is a $\rho_\alpha$-equivariant homeomorphism.
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**Theorem**

Any connected orbifold $\hat{\mathcal{O}}$ admits a connected universal covering $\rho : \hat{\mathcal{O}} \longrightarrow \mathcal{O}$. 
There are several definitions for orbifold fundamental group.

(1) Consider the based loop space of \((\mathcal{O}, \mathfrak{o})\) consisting of equivalence classes of loops in \(\mathcal{O}\), where \(\mathfrak{o}\) is a non-singular base point. For any \(k \geq 1\), the \(k\)-th homotopy group of \((\mathcal{O}, \mathfrak{o})\), denoted by \(\pi^{orb}_k(\mathcal{O}, \mathfrak{o})\), is defined to be the \((k - 1)\)-th homotopy group of the based loop space \((\Omega(\mathcal{O}, \mathfrak{o}), \mathfrak{d})\).
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2. Consider the universal covering $\rho : \tilde{\mathcal{O}} \longrightarrow \mathcal{O}$, $\pi^\text{orb}_1(\mathcal{O}, \mathfrak{o}) \cong \text{Deck}(\rho)$. 
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(2) Consider the universal covering $\rho : \hat{\mathcal{O}} \longrightarrow \mathcal{O}$, $\pi^\text{orb}_1(\mathcal{O}, \mathfrak{o}) \cong \text{Deck}(\rho)$.

(3) For a good orbifold, i.e. $G \hookrightarrow Y \twoheadrightarrow Y/G$, consider the Borel construction $Y_G \cong Y \times_G EG$, then $\pi^\text{orb}_k(Y/G) \cong \pi_k(Y_G)$.
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where $\pi_{1}^{orb}(P^{n})$ is the right-angled Coxeter group associated with $P^{n}$. Let $K$ be the dual complex of $P$.

$$\pi_{1}^{orb}(P^{n}) \cong \langle s, s \in V(K) | s^{2} = 1, (st)^{2} = 1, if (st) \in E(K) \rangle$$
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If $P^2$ is 4-gon, $\pi_1^{\text{orb}}(P^2)$ is

$$\langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = 1, ab = ba, ad = da, bc = cb, cd = dc \rangle$$
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