The combinatorial information of polytopes.

c.f. Chapter I. "Toric Topology". V.M. Buchstaber, T.E. Panov

I) Definition of convex polytopes in $\mathbb{R}^n$

Two equivalent definitions.

1. A convex polytope is the convex hull $\text{conv}(\vec{v}_1, \ldots, \vec{v}_k)$ of a finite set of points $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$.

2. A convex polyhedron $P$ is a nonempty intersection of finitely many half-spaces in $\mathbb{R}^n$:

$$P := \{ \vec{x} \in \mathbb{R}^n \mid \langle \vec{a}_i, \vec{x} \rangle + b_i \geq 0 \text{ for } i = 1, \ldots, m \}$$

where $\vec{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. A convex polytope is a bounded convex polyhedron.

E.g. 1. simplex $\Delta^n$

$$\text{conv}(\vec{e}_1, \ldots, \vec{e}_n)$$

or:

$$\begin{cases} x_i \geq 0 & i = 1, \ldots, n \\ -x_1 - \cdots - x_n + 1 \geq 0 \end{cases}$$

2. $n$-cube $I^n = [0,1]^n$

$$\text{conv}(\epsilon_1 \vec{e}_1 + \epsilon_2 \vec{e}_2 + \cdots + \epsilon_n \vec{e}_n)$$

or:

$$\{ 0 \leq x_i \leq 1. \ i = 1, \ldots, n \}$$

Assume $P^n$ is bounded.
II) face poset

A supporting hyperplane of $P$ is a affine hyperplane $H$, s.t. $H \cap P$ non empty, and $P$ lies in one side of $H$.

\[ p^n = \Delta^2 \]

The intersection $P \cap H$ is called a face of $P^n$.

- $0$-dim faces are called vertices
- $1$-dim faces are called edges
- codim-$1$ faces are called facets. (corresponding to $\langle a_i, x \rangle + b_i = 0$)

Each face is the intersection of several facets.
The faces of a given polytope $P^n$ form a partially ordered set (poset) with respect to inclusion, called face poset of $P^n$.

Note: The one skeleton of $P^n$ is a graph.

- Two polytopes are combinatorially equivalent if and only if their face posets are isomorphic
III) simple, simplicial polytopes and dual polytopes

- If exactly \( n \) facets meet at each vertex of \( P^n \), then \( P^n \) is simple.
  
  e.g. \( \Delta^n, I^n \)
  
  counterexample.
  
  octahedron.

- If each facet is a simplex, then \( P^n \) is simplicial. (come from triangulation)

Note. If \( P^n \) is simple, each \( n \)-face can be expressed as intersection of \( (n-1) \) facets uniquely.

- The polar set of a polyhedron \( P^n \subset \mathbb{R}^n \) is
  
  \( P^* := \{ \mathbf{u} \in \mathbb{R}^n \mid \langle \mathbf{u}, \mathbf{z} \rangle \leq 0 \text{ for all } \mathbf{z} \in P^n \} \)

  actually, \( \mathbf{z} \) can be replaced by vertex \( \mathbf{v}_i \).

Prop. If \( \mathbf{z} \in P^n \), \( P^* = \text{conv}(\mathbf{a}_1, \ldots, \mathbf{a}_m) \), and \((P^n)^* = P^n\).

  e.g. \((\Delta^n)^* = \Delta^n\); \((I^n)^* = \text{conv} \langle \pm e_k \rangle\)

  cross-polytope.

Thm: If \( P^n \) and \( P^* \) are dual polytopes, the face poset of \( P^* \) is obtained from face poset of \( P^n \) by reversing the inclusion relation.

  i.e.\[
  \text{vert}(P^n) \xleftarrow{1:1} \text{facets of } P^* \\
  \text{facets of } P^n \xleftarrow{1:1} \text{vert}(P^*)
  \]

Prop: \( P^n \) is simple \( \iff \) \( P^* \) is \( \Delta^n \)-simplicial. → related to simplicial complex.
IV) operations: products, hyperplane cuts and connected sums

1) products $P_1 \times P_2$. Products of two simple polytopes is also simple.

2) face truncation:

3) connected sum $\#$ at vertex.

Any 3-simple polytope can be obtained from $\Delta^3$ by finitely many face truncations.

Connected sum corresponding to connected sum of two manifolds.

$\Delta^n \# \Delta^n \rightarrow \Delta^{n+1} \times I$
v). Face vectors and Dehn-Sommerville relations.

Def. Let $P^n$ be a convex $n$-polytope. $f_i$ be the number of $i$-faces. 

- face vector (f-vector) of $P^n$. 
  $\vec{f}(P) = (f_0, f_1, \ldots, f_n)$. 
  $f_n = 1$

- $F(p) (s, t) = s^n + f_1 s^{n-1} t + \ldots + f_i s^{n-i} t^i + \ldots + f_n t^n$

- h-vector $h(P) = (h_0, h_1, \ldots, h_n)$
  $H(p) (s, t) = F(p) (s-t, t)$.

- g-vector of simple polytope $P^n$. 
  $g(P) = (g_0, g_1, \ldots, g_{n+1})$. 
  $g_1 = h_1 - h_0$. 
  $g_0 = 1$

- g-theorem: necessary and sufficient condition for simple $n$-polytope.

- g-conjecture: ... for triangulated spheres.

e.g. $\Delta^n$.

- $\vec{f}(\Delta^n) = (1, (\binom{n}{0}), \ldots, (\binom{n}{i}), 1)$

- $\vec{h}(\Delta^n) = (1, 1, \ldots, 1, 1)$

f-vector and h-vector are combinatorial invariant of polytope, but not complete.

Thm: (Dehn-Sommerville relations). The h-vector of any simple $n$-polytope is symmetric.

For small cover $(\mathbb{Z}_2)^n \to M^n \to P^n$, the $i$-th mod 2 Betti number $b_i(M^n) = h_i(P^n)$. Poincaré duality.
VI) Face ring and equivariant cohomology.

For simple polytope $P^n$ with $m$ facets, the dual polytope $P^*$ is a simplicial complex on the set $\{m_1, \ldots, m_j\}$.

The face ring (or Stanley-Reisner ring) of $P^n$ is:

$$k(P) = k[\mathbf{v}_1, \ldots, \mathbf{v}_m]/I$$

where $V_i \mapsto$ facet $F_i$ of $P^n$.

$I$ is the ideal generated by those monomials $v_i$ for which $I \subseteq \{m_j\}$ for which $\bigcap F_i = \emptyset$.

E.g.

1. $k(\Delta^n) = k[\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}]/(V_1V_3, V_2V_4, \ldots, V_{n-1}V_n)$
2. $k(I^3) = k[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]/(V_4V_5V_6)$
3. $k[P \times P'] = k[P] \otimes k[P']$

Thm (Bruns-Gubeladze) Let $k$ be a field, face ring $k(P)$ is a complete invariants of simple polytope $P$.

The Borel construction:

$$BP^n = \bigoplus_{\mathbb{Z}_+} X_{\mathbb{Z}_+}^n M^n$$

$$= (\bigoplus_{\mathbb{Z}_+} X_{\mathbb{Z}_+}^n M^n)$$

$$H^*(BP^n; \mathbb{Z}) \cong \mathbb{Z}[P^n].$$

Complex case:

$$T^n_\mathbb{Z} \xrightarrow{M^n} \mathbb{C}^n \text{ quasitopic manifold} \xrightarrow{\mathbb{Z}} H^*_\mathbb{Z}(M^n; \mathbb{Z}) \cong \mathbb{Z}[P^n].$$
For simple polytope $P^n$ with $m$ facets, assign each facet $F_i$ a vector $\lambda_i \in \mathbb{Z}^n$, s.t. if $\bigcap_{j=1}^n F_j \neq \emptyset$, \{\lambda_{ij}\} span ($\mathbb{Z}^n$).

Then we get a characteristic matrix:

$$\Lambda : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$$

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \end{pmatrix}_{n \times m}$$

We can get a small cover $M^n$ based on $(P^n, \Lambda)$ and $H^\ast (M, \mathbb{Z}) \cong \mathbb{Z}^\ast P^n/J$

compared with $H^\ast_{\mathbb{Z}^n} (M, \mathbb{Z}) \cong \mathbb{Z}^\ast P^n$.

Similar expression for complex cases.