The combinatorial information of polytopes.

c.f. chapter 1. "Toric Topology" V.M. Buchstaber, T.E Panov

I) Definition of convex polytopes in $\mathbb{R}^n$

Two equivalent definition.

① A convex polytope is the convex hull $\mathrm{conv}(\vec{v}_1, \cdots, \vec{v}_q)$ of a finite set of points $\vec{v}_1, \cdots, \vec{v}_q \in \mathbb{R}^n$.

② A convex polyhedron P is a nonempty intersection of finitely many half-spaces in $\mathbb{R}^n$:

$$P := \{ \vec{x} \in \mathbb{R}^n \mid \langle \vec{a}_i, \vec{x} \rangle + b_i \geq 0 \quad \text{for } i=1, \cdots, m\}$$

where $\vec{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. A convex polytope is a bounded convex polyhedron.

e.g. ① simplex Δ^n $\quad \mathrm{conv}(\vec{0}, \vec{e}_1, \cdots, \vec{e}_n)$

or. $\begin{cases} x_i \geq 0 & i=1, \cdots, n \\ -x_1 - \cdots - x_n + 1 \geq 0 \end{cases}$

② n-cube $I^n = [0,1]^n$ $\quad \mathrm{conv}(\epsilon_1 \vec{e}_1 + \epsilon_2 \vec{e}_2 + \cdots + \epsilon_n \vec{e}_n)$ $\quad \epsilon_i = 0, 1.$

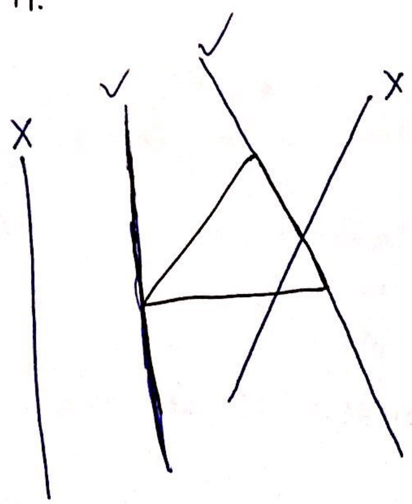
or. $\{ 0 \leq x_i \leq 1. \quad i=1, \cdots, n\}$

Assume P^n is bounded.

II) faces poset

A *supporting hyperplane* of P is a affine hyperplane H. s.t. $H \cap P$ non empty, and P lies in ~~on~~ one sides of H.

e.g. $P^n = \Delta^2$



The intersection $P \cap H$ is called a *face* of P^n.

- 0-dim faces are called *vertices*
- 1-dim faces are called *edges*
- codim-1 faces are called *facets*. (corresponding to ~~inequality~~ $\langle \vec{a}_i, \vec{x} \rangle + b_i = 0$)

Each face is the intersection of several facets.

The faces of a given polytope P^n form a partially ordered set (poset) with respect to inclusion, called *face poset* of P^n.

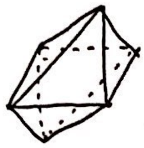
Note: The one skeleton of P^n is a graph.

- Two polytopes are *combinatorially equivalent* if and only if their face posets are isomorphic

III) simple, simplicial polytopes and dual polytopes

- If exactly n facets meet at each vertex of P^n, then P^n is simple.

  e.g. Δ^n, I^n

  counterexample.  octahedron.

- If each facet is a simplex, then P^n is simplicial. (~~come~~ from triangulation)

Note: If P^n is simple. each i-face can be expressed as intersection of $(n-i)$ facets uniquely.

- The polar set of a polyhedron $P^n \subset \mathbb{R}^n$ is (dual polytope)

$$P^* := \{ \vec{u} \in \mathbb{R}^n \mid \langle \vec{u}, \vec{x} \rangle + 1 \geq 0. \text{ for all } \vec{x} \in P^n \}$$

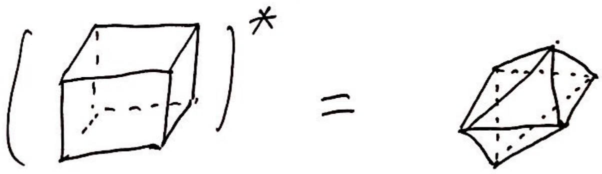
actually, $\vec{x}$ can be replaced by vertice $\{\vec{v}_1, \cdots, \vec{v}_q\}$

Prop. If $\vec{0} \in P^n$. $P^* = \mathrm{conv}(\vec{a}_1, \cdots, \vec{a}_m)$, and $(P^*)^* = P^n$.

e.g. $(\Delta^n)^* = \Delta^n$;

$(I^n)^* = \mathrm{conv}\langle \pm e_k \rangle$

cross-polytope.



Thm: If P^n and P^* are dual polytopes, the face poset of P^* is obtained from face poset of P^n by reversing the inclusion relation.

i.e. $\mathrm{vert}(P^n) \xleftrightarrow{1:1}$ facets of P^*

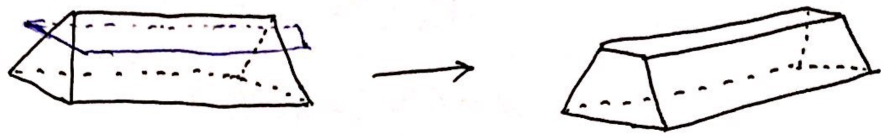
facets of $P^n \xleftrightarrow{1:1} \mathrm{vert}(P^*)$

Prop: P^n is simple $\Longleftrightarrow$ P^* is ~~simplex~~. simplicial. $\longrightarrow$ related to simplicial complex.

IV) operations: products. hyperplane cuts and connected sums

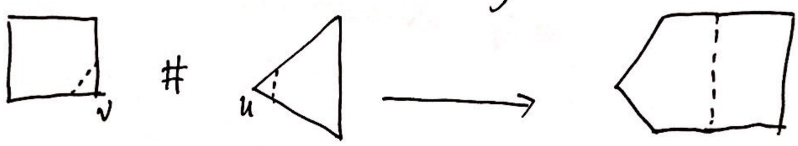
① products $P_1 \times P_2$. products of two simple polytopes is also simple.

② face truncation:



Any 3-dim simple polytope can be obtained from Δ^3 by finitely many face truncations.

③ connected sum at vertex. (corresponding to connected sum of two manifold).



$$\Delta^n \# \Delta^n \longrightarrow \Delta^{n-1} \times I$$

V). Face vectors and Dehn-Sommerville relations.

Def. Let P^n be a convex n-polytope. f_i be the number of i-faces.

face vector (f-vector) of P^n. $\vec{f}(P) = (f_0, f_1, \cdots, f_n)$. $f_n = 1$

$$F(P)(s,t) = s^n + f_{n-1}s^{n-1}t + \cdots + f_1 s t^{n-1} + f_0 t^n$$

h-vector $h(P) = (h_0, h_1, \cdots, h_n)$ $H(P)(s,t) = F(P)(s-t, t)$.

g-vector of simple polytope P^n. $\vec{g}(P) = (g_0, g_1, \cdots, g_{[\frac{n}{2}]})$. $g_i = h_i - h_{i-1}$. $g_0 = 1$

g-theorem necessary and sufficient condition for simple n-polytope.

g-conjecture ... for triangulated spheres

e.g. Δ^n.

$$\vec{f}(\Delta^n) = (1, \binom{n+1}{1}, \cdots, \binom{n+1}{n}, 1)$$

$$\vec{h}(\Delta^n) = (1, 1, \cdots, 1, 1)$$

f-vector and h-vector are combinatorial invariant of polytope, but not complete.

Thm: (Dehn-Sommerville relations). The h-vector of any simple n-polytope is symmetric i.e. $h_i = h_{n-i}$.

For small cover

$$\begin{array}{c} (\mathbb{Z}_2)^n \\ \curvearrowright \\ M^n \\ \downarrow \\ P^n \end{array}$$

The i-th mod 2 Betti number $b_i(M^n) = h_i(P^n)$. Poincaré duality.

VI) Face ring and equivariant cohomology.

For simple polytope P^n with m facets. The face poset of its dual polytope P^* is a simplicial complex on the set $[m] := \{1, \cdots, m\}$

The face ring (or Stanley-Reisner ring) of P^n is.

$$k(P) = k[v_1, \cdots, v_m]/I$$

k- coefficient ring.

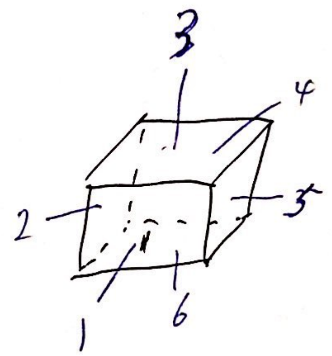
$v_i \longleftrightarrow$ facet F_i of P^n.

I is the ideal generated by those monomials v_I ~~for which~~ $I \subset [m]$. for which $\bigcap_{i \in I} F_i = \phi$.

e.g. ① $k(\Delta^n) = k[v_1, \cdots, v_{n+1}]/(v_1 v_2 \cdots v_{n+1})$

② $k(I^3) = k[v_1, v_2, \cdots, v_6]/(v_1v_4, v_2v_5, v_3v_6)$

③ $k[P_1 \times P_2] = k[P_1] \otimes k[P_2]$



Thm (Bruns-Gubeladze) Let k be a field. face ring $k(P)$ is a complete invariants of simple polytope P.

The Borel construction.

$$BP^n = E\mathbb{Z}_2^n \times_{\mathbb{Z}_2^n} M^n$$
$$= (\mathbb{R}P^\infty)^n \times_{\mathbb{Z}_2^n} M^n$$

$(\mathbb{Z}_2)^n \curvearrowright M^n$ small cover

$M^n \downarrow P^n$

$$H^*(BP^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[P^n].$$

complex case. $T^n \curvearrowright M^{2n}$ quasi-toric manifold, $M^{2n} \downarrow P^n$

$$H^*_{T^n}(M^{2n}; \mathbb{Z}) \cong \mathbb{Z}[P^n]$$

VII).

For simple polytope P^n with m facets.

assign each facet F_i a vector $\lambda_i \in \mathbb{Z}_2^n$. s.t. if $\bigcap_{j=1}^{n} F_{i_j} \neq \phi$. $\{\lambda_{i_j}\}$ span $(\mathbb{Z}_2)^n$.

Then we get a characteristic matrix.

$$\Lambda : \quad \mathcal{F} = \{F_i\}_m \longrightarrow \mathbb{Z}_2^n$$

$$\Lambda = \left(\vec{\lambda}_1, \vec{\lambda}_2, \cdots, \vec{\lambda}_m\right)_{n \times m}$$

We can get a small cover M^n based on (P^n, Λ)

and. $H^*(M, \mathbb{Z}_2) \cong \mathbb{Z}_2[P^n]/J$ J is ~~generatat~~ generated by $\Lambda \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$

compared with $H^*_{\mathbb{Z}_2^n}(M; \mathbb{Z}_2) \cong \mathbb{Z}_2[P^n]$

i.e. $\begin{cases} \lambda_{11} v_1 + \lambda_{21} v_2 + \cdots + \lambda_{m1} v_m ; \\ \quad - \quad - \quad - \quad - \quad ; \\ \lambda_{1n} v_1 + \lambda_{2n} v_2 + \cdots \lambda_{mn} v_m ; \end{cases}$

~~Sim~~ Similar expression for complex cases.